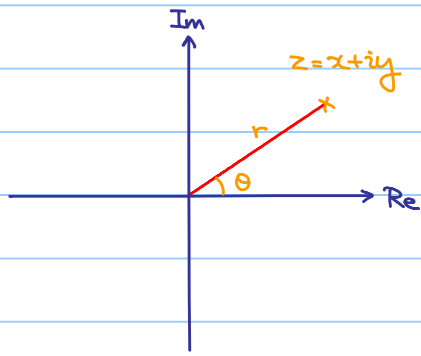
§ 4 Poincaré Disk Model

4.1 Preliminaries

$\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$ $\quad i = \sqrt{-1}$



$z = x+iy = r\cos\theta + ri\sin\theta$

If both x, y do not equal to 0:

rectangular coordinate $\longleftrightarrow$ polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \frac{y}{x} \end{cases}$$

$z \in \mathbb{R} \Leftrightarrow \theta = n\pi \quad , \; n \in \mathbb{Z}$

Conjugate : $\bar{z} = x - iy$

Modulus : $|z| = r = \sqrt{x^2+y^2}$, so $|z|^2 = x^2+y^2 = z\bar{z}$

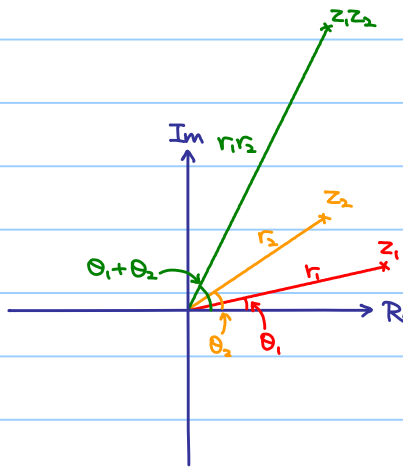
Expotential from : $e^{i\theta} = \cos\theta + i\sin\theta$

$z = re^{i\theta}$

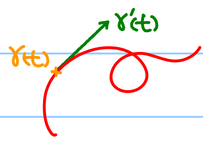
Let $z_1 = r_1e^{i\theta_1}$, $z_2 = r_2e^{i\theta_2}$,

then $z_1z_2 = (r_1e^{i\theta_1})\cdot(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}$

$$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$



Curve on $\mathbb{C}$. $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{C}$, $\gamma(t) = x(t) + iy(t)$

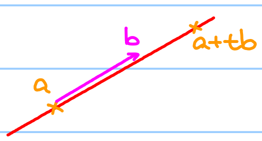


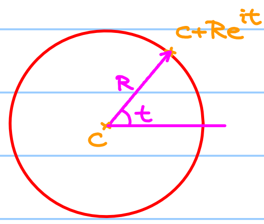
Let $\gamma_j(t) = x_j(t) + iy_j(t)$, $-\varepsilon < t < \varepsilon$, $j = 1, 2$, be two curves on $\mathbb{C}$ such that $\gamma_1(0) = \gamma_2(0) = z_0$.

Let θ be the angle between γ_1 and γ_2 at z_0. Then $e^{i\theta} = \dfrac{\frac{\gamma_1'}{|\gamma_1'|}}{\frac{\gamma_2'}{|\gamma_2'|}}$.

Line . $\gamma(t) = a + tb$ where $a, b \in \mathbb{C}$, $t \in \mathbb{R}$

Circle : $\gamma(t) = c + Re^{it}$ where $c \in \mathbb{C}$, $R > 0$, $t \in (-\pi, \pi]$





Definition 4.1.1

Let $f : \mathbb{C} \to \mathbb{C}$

Translation (Translated by $a \in \mathbb{C}$) : $f(z) = z + a$

Rotation (Rotate by θ in anti-clockwise direction) : $f(z) = e^{i\theta} z$

Scaling (Scale by a factor $r \in \mathbb{R}$, $r \neq 0$) : $f(z) = rz$

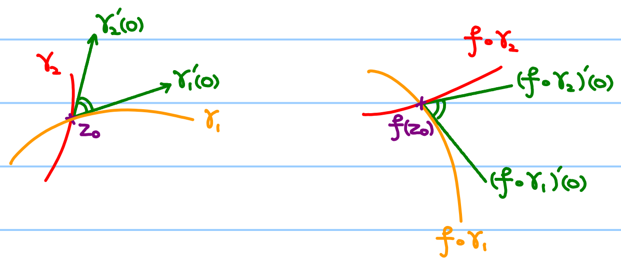
Reciprocation : $f(z) = \frac{1}{z}$

Exercise 4.1.1

Show that translation, rotation, scaling and reciprocation are angle preserving maps,

i.e. if γ_1 and γ_2 are two curves meet at $\gamma_1(0) = \gamma_2(0) = z_0$, then

angle between γ_1 and γ_2 at z_0 = angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(z_0)$

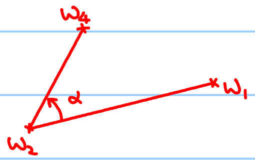


Definition 4.1.2

Let w_1, w_2, w_3, w_4 be four distinct complex numbers,

then the four point ratio is defined by $[w_1, w_2, w_3, w_4] = \frac{w_4 - w_2}{w_1 - w_2} \Big/ \frac{w_4 - w_3}{w_1 - w_3} = \frac{w_4 - w_2}{w_1 - w_2} \cdot \frac{w_1 - w_3}{w_4 - w_3}$

If $\frac{w_4 - w_2}{w_1 - w_2} = R e^{i\alpha}$



α : (oriented) angle $\angle w_1 w_2 w_4$

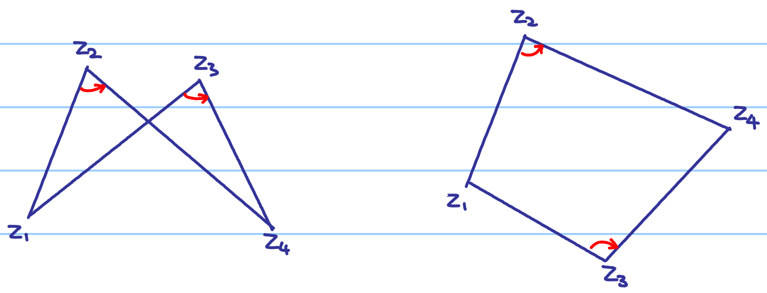
R : ratio between two sides.

Proposition 4.1.1

$[w_1, w_2, w_3, w_4] \in \mathbb{R}$ if and only if w_1, w_2, w_3, w_4 are concyclic or collinear.

proof :

$[w_1, w_2, w_3, w_4] \in \mathbb{R} \iff \angle w_1 w_2 w_4 - \angle w_1 w_3 w_4 = n\pi \iff w_1, w_2, w_3, w_4$ are concyclic or collinear.



.. Any other cases ?

Therefore, given three noncollinear points w_1, w_2, w_3, the equation of the circle passing through w_1, w_2, w_3 can be given by $[w_1, w_2, w_3, z] \in \mathbb{R}$ or $\operatorname{Im}[w_1, w_2, w_3, z] = 0$

Given three collinear points w_1, w_2, w_3,

$$[w_1, w_2, w_3, z] = \frac{z - w_2}{w_1 - w_2} \cdot \frac{w_1 - w_3}{z - w_3} = \frac{z - w_2}{z - w_3} \cdot \underbrace{\frac{w_3 - w_1}{w_2 - w_1}}_{\in \mathbb{R}}$$

$\therefore [w_1, w_2, w_3, z] \in \mathbb{R} \Leftrightarrow \frac{z - w_2}{z - w_3} \in \mathbb{R}$

the equation of the straight line passing through (z_1), z_2 and z_3 can be given by

$\frac{z - w_2}{z - w_3} \in \mathbb{R}$ or $\operatorname{Im} \frac{z - w_2}{z - w_3} = 0$

Exercise 4.1.2

If $[w_1, w_2, w_3, w_4] \in \mathbb{R}$, then all 24 four point ratios given by permuting w_1, w_2, w_3, w_4 are real as well.

Proposition 4.1.2

Four point ratio is preserved by translation, rotation, scaling and reciprocation i.e.

$[f(w_1), f(w_2), f(w_3), f(w_4)] = [w_1, w_2, w_3, w_4]$ if f is a translation, a rotation, a scaling or reciprocation.

Direct consequence:

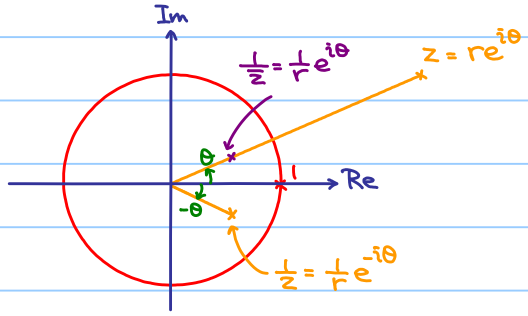
Proposition 4.1.3

Translation, rotation, scaling and reciprocation map a circle or a line to a circle or a line.

Exercise 4.1.3

If $f(z) = \frac{1}{\bar{z}} = \frac{z}{|z|^2}$ (inversion), show that $[f(w_1), f(w_2), f(w_3), f(w_4)] = \overline{[w_1, w_2, w_3, w_4]}$.

Hence inversion maps a circle or a line to a circle or a line.



Definition 4.1.3

A Möbius transformation is a function $f: \mathbb{C} \to \mathbb{C}$ in form of

$f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad-bc \neq 0$.

Exercise 4.1.4

Let $f_1(z) = z + \frac{d}{c}$, $f_2(z) = \frac{1}{z}$, $f_3(z) = \frac{bc-ad}{c^2} z$, $f_4(z) = z + \frac{a}{c}$

Show that $(f_4 \circ f_3 \circ f_2 \circ f_1)(z) = \frac{az+b}{cz+d}$, i.e. every Möbius transformation can be expressed as a composition of translation, rotation, scaling and reciprocation.

Hence, Möbius transformations are bijective, preserving angles and four point ratios and they map a circle or a line to a circle or a line.

Also show that $f(z) = (f_1^{-1} \circ f_2^{-1} \circ f_3^{-1} \circ f_4^{-1})(z) = \frac{dz-b}{-cz+a}$ which is again a Möbius transformation.

Now, we consider a particular class of Möbius transformation.

Exercise 4.1.3

Let $a, \lambda \in \mathbb{C}$ with $|a| < 1$, $|\lambda| = 1$ and let $f(z) = \lambda \frac{z-a}{\bar{a}z-1}$ (Note $-1 + a\bar{a} \neq 0$). Show that

(i) if $|z| < 1$, then $|f(z)| < 1$; if $|z| = 1$, then $|f(z)| = 1$

(ii) $f^{-1}(z)$ is also in the form $\lambda' \frac{z-a'}{\bar{a}'z-1}$ where $a', \lambda' \in \mathbb{C}$ with $|a'| < 1$, $|\lambda'| = 1$.

(Ans. $f^{-1}(z) = \frac{1}{\lambda} \frac{z-\lambda a}{(\overline{\lambda a})z-1}$)

Note if $w = f(z)$, $z = f^{-1}(w)$,

then by (i) and (ii), we have $|z| = 1$ if and only if $|w| = |f(z)| = 1$.

$|z| < 1$ if and only if $|w| = |f(z)| < 1$.

Hence, f maps the unit circle to itself and maps the open unit disk to itself.

Theorem 4.1.1

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The set of all biholomorphic function $f: \mathbb{D} \to \mathbb{D}$

$\mathrm{Aut}(\mathbb{D}) = \left\{ f(z) = \lambda \frac{z-a}{\bar{a}z-1} : a, \lambda \in \mathbb{C}, |a| < 1, |\lambda| = 1 \right\}$.

Let $w = \frac{z-a}{\bar{a}z-1}$. (Take $\lambda = 1$)

$\bar{a}wz - w = z - a$

$(\bar{a}w - 1)z = w - a$

$$z = \frac{w-a}{\bar{a}w-1} = \frac{(w-a)(a\bar{w}-1)}{|\bar{a}w-1|^2}$$

Let $z = x+iy$, $w = u+iv$, $a = c+id$

$$x+iy = \frac{[(u-c)+i(v-d)][(cu+dv-1)+i(du-cv)]}{|(cu+dv-1)+i(cv-du)|^2}$$

$$= \frac{(u-c)(cu+dv-1)-(v-d)(du-cv)}{(cu+dv-1)^2+(cv-du)^2} + i\,\frac{(u-c)(du-cv)+(v-d)(cu+dv-1)}{(cu+dv-1)^2+(cv-du)^2}$$

$$x = \frac{cu^2+cv^2+(d^2-c^2-1)u-2cdv+c}{(c^2+d^2)u^2+(c^2+d^2)v^2-2cu-2dv+1} \qquad y = \frac{du^2+dv^2-2cdu+(c^2-d^2-1)v+d}{(c^2+d^2)u^2+(c^2+d^2)v^2-2cu-2dv+1}$$

When $x = x_0$, we have $[c-x_0(c^2+d^2)](u^2+v^2)+(d^2-c^2-1+2cx_0)u+(2dx_0-2cd)v+(c-x_0)=0$

When $y = y_0$, we have $[d-y_0(c^2+d^2)](u^2+v^2)+(2cy_0-2cd)u+(c^2-d^2-1+2dy_0)v+(d-y_0)=0$

circles or straight lines

i.e. f maps vertical and horizontal lines to circles or straight lines.

Example 4.1.1

Let $f(z) = \frac{z-(\frac{1}{2}+\frac{i}{2})}{(\frac{1}{2}-\frac{1}{2}i)z-1}$ i.e. $\lambda = 1$, $a = c+di = \frac{1}{2}+\frac{1}{2}i$

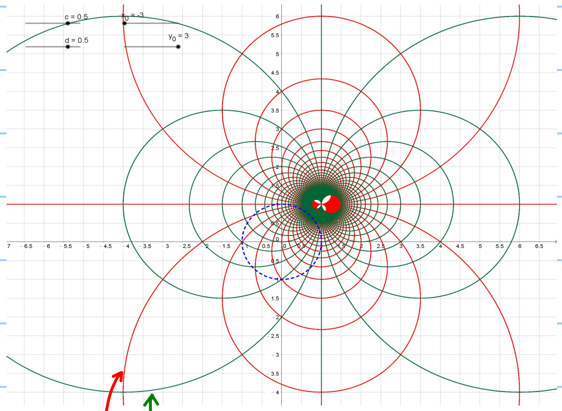
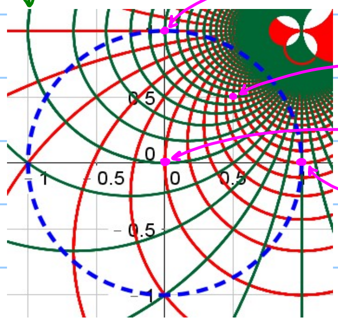


Image of $y = y_0$ under f

Image of $x = x_0$ under f

Image of $y = \frac{1}{2}$ under f



$f(1) = i$

$f(0) = \frac{1}{2}+\frac{1}{2}i$

$f(\frac{1}{2}+\frac{1}{2}i) = 0$

$f(i) = 1$

Image of $x = \frac{1}{2}$ under f

Green curves and red curves are perpendicular

Results related to inversion.

Let $f(z) = \frac{1}{\bar{z}}$

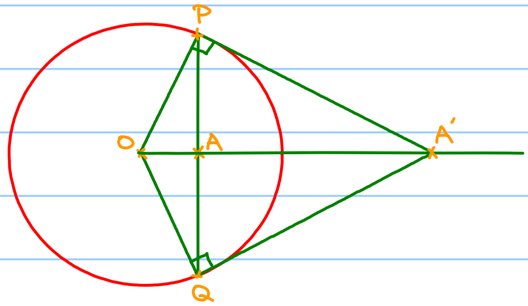
Let Γ be the unit circle on $\mathbb{C}$, i.e. $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$.

Proposition 4.1.4

Let A be a point inside Γ. Let PQ be the chord such that $PQ \perp OA$.

Then the tangents to Γ at P and Q meets the ray r_{OA} at the point A′ which is exactly the inverse of A.

proof:



$\Delta OAP \sim \Delta OPA'$

$$\frac{OA}{OP} = \frac{OP}{OA'}$$

$$OA' = \frac{1}{OA}$$

Proposition 4.1.5

		inversion	
(i)	line passing through O	$\longleftrightarrow$	itself
(ii)	line not passing through O		circle passing through O
(iii)	circle not passing through O		circle not passing through O
(iv)	circle perpendicular to Γ		itself

Proposition 4.1.6

If a circle Γ' contains a point z and its inverse $f(z) = \frac{1}{\bar{z}}$, then Γ' maps to itself under inversion.

Hence Γ' is perpendicular to Γ.

4.2 Poincaré Disk Model

Poincaré Disk : $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with Poincaré metric $ds^2 = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2} = \frac{4dr^2}{(1-r^2)^2} = \frac{4dzd\bar{z}}{(1-|z|^2)^2}$

Proposition 4.2.1

Let $z_1, z_2 \in \mathbb{D}$, there exists unique circle or line γ passing through z_1, z_2 such that γ is perpendicular to $\mathcal{T}$, where $\mathcal{T} = \{z \in \mathbb{C} : |z| = 1\}$.

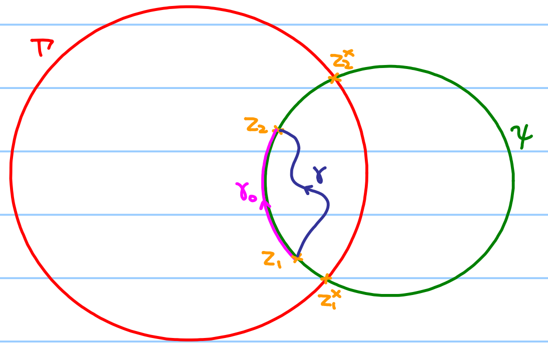
proof:

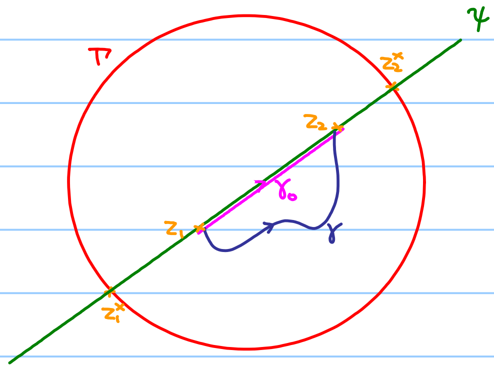
If z_1 and z_2 lie on a diameter of $\mathcal{T}$, then let γ be the unique straight line containing z_1 and z_2.

Otherwise, let γ be the unique circle passing through $z_1, z_2, \frac{1}{\bar{z}_1}$.

Let $\ell = \mathbb{D} \cap \gamma$ and $\gamma_0 \subseteq \mathcal{T}$ which is the arc with endpoints z_1 and z_2.

Let $\mathcal{T} \cap \gamma = \{z_1^*, z_2^*\}$ where they are named so that we can go along γ according the order z_1^*, z_1, z_2, z_2^*.





Theorem 4.2.1

(i) Let $\gamma \subseteq \mathbb{D}$ be any curve with endpoints z_1 and z_2. Then

Length of $\gamma = \int_\gamma ds \geq \int_{\gamma_0} ds =$ Length of γ_0 (with respect to Poincaré metric)

(ii) Length of $\gamma_0 = \ln |[z_1^*, z_1, z_2, z_2^*]| = \ln \left| \frac{z_2^* - z_1}{z_1^* - z_1} \Big/ \frac{z_2^* - z_2}{z_1^* - z_2} \right|$

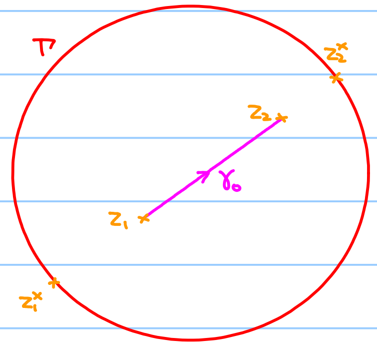
Therefore we define a distance function $d : \mathbb{D} \times \mathbb{D} \to \mathbb{R}^+$ by $d(z_1, z_2) = \ln |[z_1^*, z_1, z_2, z_2^*]|$ and γ is said to be a line (geodesic) of $\mathbb{D}$.

We call a line on $\mathbb{D}$ to be a P-line to distinguish it from an ordinary straight line on $\mathbb{C}$.

Let $z_1 = R_1 e^{i\alpha}$ and $z_2 = R_2 e^{i\alpha}$ where $-1 < R_1 < R_2 < 1$

Then $z_1^* = -e^{i\alpha}$ and $z_2^* = e^{i\alpha}$ and $\gamma_0(t) = te^{i\alpha}$, $t \in [R_1, R_2]$, is the P-line segment joining z_1 and z_2.

In polar coordinates, $\gamma_0(t) = (r(t), \theta(t)) = (t, \alpha)$



Then
$$
\begin{aligned}
d(z_1,z_2) &= \int_{\gamma_0} ds \\
&= \int_{R_1}^{R_2} \frac{2}{1-r^2}\,dr \\
&= \int_{R_1}^{R_2} \frac{1}{1+r} + \frac{1}{1-r}\,dr \\
&= \big[\ln|1+r| - \ln|1-r|\big]_{R_1}^{R_2} \\
&= \ln\left|\frac{1-R_1}{1+R_1}\,\frac{1+R_2}{1-R_2}\right| \\
&= \ln\left|\frac{e^{i\alpha}-R_1e^{i\alpha}}{-e^{i\alpha}-R_1e^{i\alpha}} \Big/ \frac{e^{i\alpha}-R_2e^{i\alpha}}{-e^{i\alpha}-R_2e^{i\alpha}}\right| \\
&= \ln\left|[z_1^*, z_1, z_2, z_2^*]\right|
\end{aligned}
$$

In particular, $d(0,z) = \ln\frac{1+r}{1-r}$ where $r = |z|$.

Note : z_1^*, z_1, z_2, z_2^* lie on a line or a circle γ

$\Rightarrow f(z_1^*), f(z_1), f(z_2), f(z_2^*)$ lie on a line or a circle $f(\gamma)$

Also $|z_1^*| = |z_2^*| = 1 \Rightarrow |f(z_1^*)| = |f(z_2^*)| = 1$ i.e. $f(\gamma) \cap T = \{f(z_1^*), f(z_2^*)\}$

$\therefore f(z_1)^* = f(z_1^*)$ and $f(z_2)^* = f(z_2^*)$

Direct consequence :

Theorem 4.2.2

If $f \in Aut(\mathbb{D})$, i.e. $f: \mathbb{D} \to \mathbb{D}$ defined by $f(z) = \lambda \frac{z-a}{\bar{a}z-1}$ where $a, \lambda \in \mathbb{C}$, $|a|<1$, $|\lambda| = 1$.

then $d(f(z_1), f(z_2)) = d(z_1, z_2)$ for all $z_1, z_2 \in \mathbb{D}$

proof:

$d(f(z_1), f(z_2)) = \ln|[f(z_1)^*, f(z_1), f(z_2), f(z_2)^*]| = \ln|[f(z_1^*), f(z_1), f(z_2), f(z_2^*)]| = \ln|[z_1^*, z_1, z_2, z_2^*]| = d(z_1, z_2)$

Therefore, f is length and angle preserving for all $f \in Aut(\mathbb{D})$.

Example 4.2.1

Let $z_1 = \frac{1}{2}$, $z_2 = \frac{1}{2}i$. Find $d(z_1, z_2)$.

Question: How to find z_1^* and z_2^* ?

Let $f(z) = \frac{z - \frac{1}{2}}{\frac{1}{2}z - 1} = \frac{2z-1}{z-2} \in \mathrm{Aut}(\mathbb{D})$, (i.e. take $\lambda = 1$, $a = z_1$)

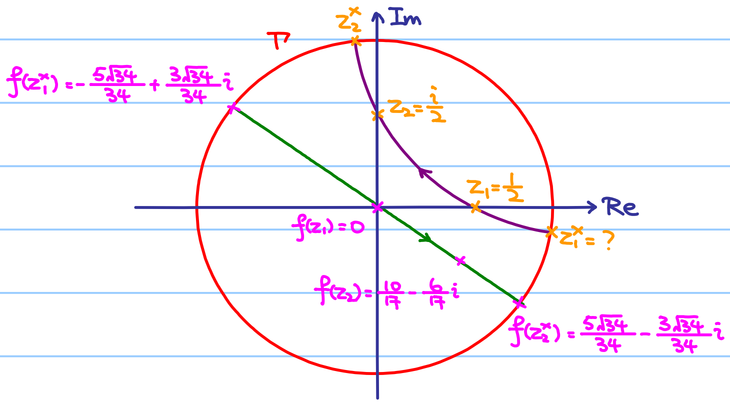
Then $f(z_1) = 0$, $f(z_2) = \frac{10}{17} - \frac{6}{17}i$

$d(z_1, z_2) = d(f(z_1), f(z_2)) = d(0, f(z_2)) = \ln \frac{1+|f(z_2)|}{1-|f(z_2)|} = \ln \frac{17+2\sqrt{34}}{17-2\sqrt{34}} \approx 1.6807$

Check: $f^{-1}(z) = \frac{2z-1}{z-2}$

$\therefore z_1^* = f^{-1}(f(z_1^*)) = \frac{4+\sqrt{34}}{10} + \frac{4-\sqrt{34}}{10}i$

$z_2^* = f^{-1}(f(z_2^*)) = \frac{4-\sqrt{34}}{10} + \frac{4+\sqrt{34}}{10}i$



Example 4.2.2

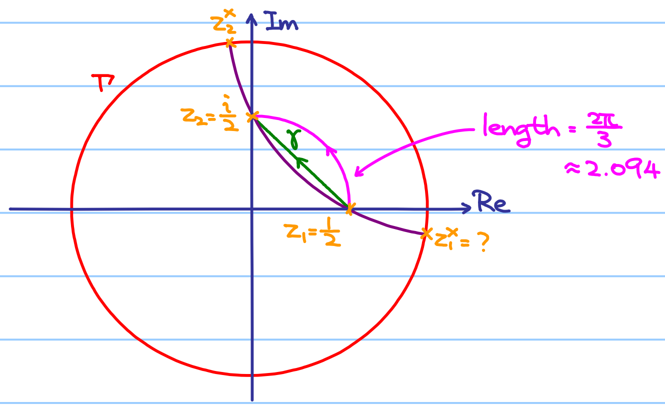
Let $\gamma(t) = \frac{1}{2}(1-t) + \frac{1}{2}ti$, $t \in [0,1]$. Then $x = \frac{1}{2}(1-t)$, $y = \frac{1}{2}t$, $z_1 = \gamma(0) = \frac{1}{2}$, $z_2 = \gamma(1) = \frac{1}{2}i$.

$x^2 + y^2 = \frac{1}{4}(2t^2 - 2t + 1)$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1}{2}$

$$\int_\gamma ds = \int_\gamma \frac{2}{1-x^2-y^2}\sqrt{dx^2+dy^2}$$
$$= \int_0^1 \frac{2}{1-\frac{1}{4}(2t^2-2t+1)}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
$$= \int_0^1 \frac{4\sqrt{2}}{-2t^2+2t+3}\,dt$$
$$= \frac{4\sqrt{14}}{7}\ln\left(\frac{\sqrt{7}+1}{\sqrt{7}-1}\right) \approx 1.701$$

Note: $\int_\gamma ds > d(z_1, z_2)$



Exercise 4.2.1

Show that the length of the circle

$(r(t), \theta(t)) = (R, t)$, $t \in [-\pi, \pi]$, $R \in (0,1)$ is $\frac{4\pi R}{1-R^2}$